INTERIOR OPERATORS AND TOPOLOGICAL CATEGORIES

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Abstract. The introduction of the categorical notion of closure operators has unified various important notions and has led to interesting examples and applications in diverse areas of mathematics (see for example, Dikranjan and Tholen ([5])). For a topological space it is well-known that the associated closure and interior operators provide equivalent descriptions of the topology, but this is not true in general. So, it makes sense to define and study the notion of interior operators I in the context of a category $\mathfrak{C}$ and a fixed class $\mathcal{M}$ of monomorphisms in $\mathfrak{C}$ closed under composition in such a way that $\mathfrak{C}$ is finitely $\mathcal{M}$-complete and the inverse images of morphisms have both left and right adjoint, which is the purpose of this paper.
Then we construct a concrete category $\mathfrak{C}_I$ over $\mathfrak{C}$ which is a topological category. Furthermore, we provide some examples and discuss some of their properties: Kuratowski interior operator, Grothendieck interior operator, interior operators on Grothendieck topos and interior operators on the category of fuzzy topological spaces.

0. Introduction

Kuratowski operators (closure, interior, exterior, boundary and others) have been used intensively in General Topology ([12], [13]). Category Theory provides a variety of notions which expand on the lattice-theoretic concept of interior operator ([6]): For a topological space it is well-known that the associated closure and interior operators provide equivalent descriptions of the topology; but it is not generally true in other categories, consequently it makes sense to define and study the notion of interior operators I in the

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context of a category $\mathfrak{C}$ and a fixed class $\mathcal{M}$ of monomorphisms in $\mathfrak{C}$ closed under composition in such a way that $\mathfrak{C}$ is finitely $\mathcal{M}$-complete and the inverse images of morphisms have both left and right adjoint.

The paper is organized as follows: Following ([5]) we introduce, in section 1, the basic categorial framework on subobjects, inveres images and image factorization as needed throughout the paper. In section 2, we present the concept of interior operator I for suitable categories and then we construct a topological category $(\mathfrak{C}_I, U)$. Finally in section 3 we provide some examples and discuss some of their properties: Kuratowski interior operator, Grothendieck interior operator, interior operators on Grothendieck topos and interior operators on the category of fuzzy topological spaces.

1. Preliminaries on Subobjects, Inverse Images and its adjoints

In this section we provide the basic categorial framework on subobjects, inveres images and image factorization as needed throughout the paper.

1.1. **$\mathcal{M}$-subobjects.** In order to allow for sufficient flexibility, as in Dikrajan and Tholen [5], we consider a category $\mathfrak{C}$ and a fixed class $\mathcal{M}$ of monomorphisms in $\mathfrak{C}$ which will play the role of subobjects. We assume that

- $\mathcal{M}$ is closed under composition with isomorphisms.
- $\mathcal{M}$ contains al identity morphisms.

For every object X in $\mathfrak{C}$, let $\mathcal{M}/X$ the class of all $\mathcal{M}$-morphisms with codomain X; the relation given by

$$m \leqslant n \Leftrightarrow (\exists j) \ \ n \circ j = m$$

$$
\begin{tikzcd}
M \arrow[rr, "j"] \arrow[dr, "m"'] & & N \arrow[dl, "n"] \\
& X &
\end{tikzcd}
\tag{1}
$$

is reflexive and transitive, hence $\mathcal{M}/X$ is a preordered class. Since n is monic, the morphism j is uniquely determined, and it is an isomorphism of $\mathfrak{C}$ if and only if $n \leqslant m$ holds; in this case m and n are isomorphic, and we write $m \cong n$. Of course, $\cong$ is an equivalence relation, and $\mathcal{M}/X$ modulo $\cong$ is a partially ordered class for which we can use all lattice-theoretic terminology. If $\widehat{m}$ denotes de $\cong$-equivalence class of m, we have, in particular, the equivalence

$$m \cong n \Leftrightarrow \widehat{m} = \widehat{n}.$$

From now on $\widehat{\mathcal{M}/X}$ denotes the partially orderd class $\mathcal{M}/X$ modulo $\cong$, and m denotes the class $\widehat{m}$.

1.2. **Inverse images are $\mathcal{M}$-pullbacks.** For our fixed class $\mathcal{M}$ of monomorphisms in the category $\mathfrak{C}$, we say that $\mathfrak{C}$ has $\mathcal{M}$-pullbacks if, for every morphism $F : X \to Y$ and $n \in \widehat{\mathcal{M}/Y}$, a pullback diagram

$$
\begin{tikzcd}
M \arrow[r, "f'"] \arrow[d, "m"'] & N \arrow[d, "n"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
\tag{2}
$$

exists, with $m \in \widehat{\mathcal{M}/X}$. Of course, as an $\mathcal{M}$-subobject of $\mathfrak{C}$, m is uniquely determined; it is called **the inverse image of n under f** and denoted by $f^{-1}(n) : f^{-1}(N) \to X$. The pullback property of (2) yields that

$$f^{-1}(-) : \widehat{\mathcal{M}/Y} \to \widehat{\mathcal{M}/X}$$

is an order-preserving map so that

$$k \leqslant n \Rightarrow f^{-1}(k) \leqslant f^{-1}(n)$$

holds.

$$\begin{tikzcd}
& f^{-1}(N) \arrow[r] \arrow[ddl] & N \arrow[dd, "n"] \\
f[-1(K) \arrow[ur] \arrow[rr, "f^{-1}(n)"'] \arrow[d, "f^{-1}(k)"'] & & K \arrow[u, "j"] \arrow[d, "k"] \\
X \arrow[rr, "f"'] & & Y
\end{tikzcd} \tag{3}$$

1.3. **When the subobjects form a large-complete lattice.** If $\mathfrak{C}$ has $\mathcal{M}$-pullbacks and if $\mathcal{M}$ is closed under composition, the ordered class $\widehat{\mathcal{M}/X}$ has binary meets for every object X: one obtains the meet

$$m \wedge n : M \wedge N \to X$$

as the diagonal of the pullback diagram

$$\begin{tikzcd}
M \wedge N \arrow[r] \arrow[d] & N \arrow[d, "n"] \\
M \arrow[r, "m"] & X
\end{tikzcd} \tag{4}$$

In general, for any $\mathcal{M}$, we say that $\mathfrak{C}$ has **$\mathcal{M}$-intersections** if for every family $(m_i)_{i \in I}$ in $\widehat{\mathcal{M}/X}$ (I may be a proper class or empty) a multiple pullback diagram

$$\begin{tikzcd}
& M_i \arrow[dl, "j_i"'] \arrow[dr, "m_i"] & \\
M \arrow[rr, "m"] & & X
\end{tikzcd} \tag{5}$$

exists in $\mathfrak{C}$ with $m \in \widehat{\mathcal{M}/X}$. One easily verifies that m indeed assumes the role of the meet of $(m_i)_{i \,\in\, I}$ in $\widehat{\mathcal{M}/X}$. Hence we writes

$$m = \bigwedge_{i \in I} m_i : \bigwedge_{i \in I} M_i \to X.$$

Proposition 1.1. *If $\mathfrak{C}$ has $\mathcal{M}$-intersections then every ordered class $\widehat{\mathcal{M}/X}$ has the structure of a large-complete lattice, i.e. class-indexed meets and joins exist in $\widehat{\mathcal{M}/X}$, for every object $X \in \mathfrak{C}$.*

Proof. As usual, we construct the join of $(m_i)_{i \,\in\, I}$ in $\widehat{\mathcal{M}/X}$ as the meet of all upper bounds of $(m_i)_{i \,\in\, I}$ in $\widehat{\mathcal{M}/X}$. ■

If $\mathfrak{C}$ has also $\mathcal{M}$-pullbacks, it is easy to see that the join $m \in \widehat{\mathcal{M}/X}$ of $(m_i)_{i \,\in\, I}$ has the following categorical property: there are morphisms j_i, $i \in I$, such that

(1) $m \cdot j_i = m_i$, for all $i \in I$;

(2) whenever we have commutative diagrams

$$\begin{tikzcd} M_i \arrow[r, "u_i"] \arrow[d, "j_i"'] & N \arrow[dd, "n"] \\ M \arrow[d, "m"'] & \\ X \arrow[r, "v"] & Z \end{tikzcd} \tag{6}$$

in $\mathfrak{C}$ with $m \in \mathcal{M}$, then there is a uniquely determined morphism $\omega : M \to N$ with $n \cdot \omega = v \cdot m$, and $\omega \cdot j_i = u_i$, for all $i \in I$.

A subobject $m \in \widehat{\mathcal{M}/X}$ is called an **$\mathcal{M}$-union** of $(m_i)_{i \,\in\, I}$ if this categorical property holds. Letting $v = 1_X$ in (6) we see that unions are joins in $\widehat{\mathcal{M}/X}$,

hence we writes

$$m = \bigvee_{i\in I} m_i : \bigvee_{i\in I} M_i \to X.$$

When $I = \emptyset$, the union $\bigvee_{i\in I} m_i$ (if it exists) is called the **trivial $\mathcal{M}$-subobject of** X; it is the least element of $\widehat{\mathcal{M}/X}$ and therefore denoted by $o_X : O_X \to X$.

Its characteristic categorical property (c.f. Diagram (6)) reads as follows: for every diagram

$$\begin{tikzcd} O_X \arrow[d, "o_X"'] & N \arrow[d, "n"] \\ X \arrow[r, "v"] & Z \end{tikzcd} \tag{7}$$

with $n \in \mathcal{M}$ there is a unequely determined morphism $\omega : O_X \to N$ with $n \centerdot \omega = v \centerdot o_X$.

Note that if the category $\mathfrak{C}$ has initial object I, then o_X is the $\mathcal{M}$-part of the right $\mathcal{M}$-factorization of the only morphism $I \to X$. This is equivalent to the existence of "solution-set conditions" (c. f. [8], I.4 or [9], V.6)

1.4. **Review of pairs of adjoint maps.** Images of subobjects are given by left-adjoints of the maps $f^{-1}(-)$. We remember that a pair of mappings $\phi : P \to Q$ and $\psi : Q \to P$ between preordered classes P, Q are adjoint if

$$\phi(m) \leqslant n \Leftrightarrow m \leqslant \psi(n) \tag{8}$$

holds for all $m \in P$ and $n \in Q$, in which case one says that ϕ is left-adjoint of ψ or ψ is right-adjoint of ϕ and we writes $\phi \vdash \psi$. Note that adjoints determine each other uniquely, up to the equivalence relation given by ($x \cong y \Leftrightarrow x \leqslant y$ and $y \leqslant x$). In other words, in ordered classes adjoints determine each other uniquely.

Lemma 1.2. *The following assertions are equivalent for any pair of mappings $\phi : P \to Q$ and $\psi : Q \to P$ between large-complete lattices:*

(i) $\phi \vdash \psi$;

(ii) ϕ *is order-preserving, and* $\phi(m) = \bigwedge\{n \in Q \mid m \leqslant \psi(n)\}$ *holds for all* $m \in P$;

(iii) ψ *is order-preserving, and* $\psi(n) = \bigvee\{m \in P \mid \phi(m) \leqslant n\}$ *holds for all* $n \in Q$;

(iv) ϕ *and* ψ *are order-preserving, and*

$$m \leqslant \psi(\phi(m)) \quad \text{and} \quad \phi(\psi(n)) \leqslant n$$

holds for all $m \in P$ *and* $n \in Q$.

Proof. *(i)* $\to$ *(ii)* & *(iii)* Putting $n = \phi(m)$ in (8), we obtain $m \leqslant \psi(\phi(m))$, hence $\phi(m) \in Q_m$,, where $Q_m = \{n \in Q \mid m \leqslant \psi(n)\}$, Furthermore, for all $n \in Q_m$ (8) yields $\phi(m) \leqslant n$, hence $\phi(m) = \bigwedge Q_m$. This formula implies that ϕ is order-preserving. Dually we obtain the formula for ψ as given in *(iii)*, and that ψ is order-preserving.

(ii) $\to$ *(iv)* As mentioned before, the given formula for ϕ implies its monotonicity. Furthermore, since $\phi(m) \in Q_m$, we have $m \leqslant \psi(\phi(m))$, and since $n \in Q_{\psi(n)}$, we have $\phi(\psi(n)) \leqslant n$ for all $m \in P$ and $n \in Q$.

(iii) $\to$ *(iv)* follows dually.

(iv) $\to$ *(i)* $m \leqslant \psi(n)$ implies $\phi(m) \leqslant \phi(\psi(n)) \leqslant n$, and $\phi(m) \leqslant n$ implies $m \leqslant \psi(\phi(m)) \leqslant \psi(n)$. ■

The most important property of adjoints pairs is the preservation of joins and meets

Proposition 1.3. *If $\phi \vdash \psi$, where $\phi : P \to Q$ and $\psi : Q \to P$ are mappings between large-complete lattices, then ϕ preserves all joins and ψ preserves*

all meets. Hence we have the formulas

$$\phi\left(\bigvee_{i\in I} m_i\right) = \bigvee_{i\in I}\phi\left(m_i\right) \quad \textit{and} \quad \psi\left(\bigwedge_{i\in I} n_i\right) = \bigwedge_{i\in I}\psi\left(n_i\right).$$

Furthermore, $\phi\,.\,\psi\,.\,\phi = \phi$ and $\psi\,.\,\phi\,.\,\psi = \psi$, so that ϕ and ψ give a biyective correspondence between $\phi(P)$ and $\psi(Q)$.

Proof. By monotonicity of ϕ, $\phi(m)$ is an upper bound of $\{\phi(m_i) \mid i \in I\}$, with $m = \bigvee_{i\in I} m_i$. For any other upper bound n, we have $m_i \leqslant \psi(n)$ for all $i \in I$ by (8), hence $m \leqslant \psi(n)$. Application of (8) again yields $\phi(m) \leqslant n$. This proves that ϕ preserves joins. The assertion for ψ follows dually.
Furthermore, when applying the order-preserving map ϕ to the first inequality of (*iv*) in the Lemma 1.2, we obtain $\phi(m) \leqslant \phi(\psi(\phi(m)))$, and when exploting the second inequality in case $n = \phi(m)$, we obtain $\phi(\psi(\phi(m))) \leqslant \phi(m)$. Hence $\phi\,.\,\psi\,.\,\phi = \phi$ and $\psi\,.\,\phi\,.\,\psi = \psi$ follows dually. ■

The converse of the first statement of Proposition 1.3 holds as well

Theorem 1.4. *Let P, Q be partially ordered sets, then*

1. *If an order-preserving map $\psi : Q \to P$ has left adjoint $\phi : P \to Q$, ψ preserve all meets which exist in Q.*
2. *If Q has all meets and ψ preserves then, ψ has a left adjoint.*
3. *If an order-preserving map $\phi : P \to Q$ has right adjoint $\psi : Q \to P$, ϕ preserve all joins which exist in P.*
4. *If P has all joins and ϕ preserves then, ϕ has a right adjoint.*

Proof.

It suffices to show (1) and (2) since (3) and (4) follows by dualization.

(1) Let X a subset of Q such that $\bigwedge X$ exists. Since ψ is order-preserving,

$\psi\left(\bigwedge X\right)$ is a lower bound of $\{\psi(x) \mid x \in X\}$. But if p is any lower bound for this set, then we have $p \leqslant \psi(x)$ for all $x \in X$, whence $\phi(p) \leqslant x$ for all $x \in X$, so $\phi(p) \leqslant \bigwedge X$ and $p \leqslant \psi(\bigwedge X)$.

(2) By definition of an adjoint, $\phi(p)$ most be the smallest $q \in Q$ satisfying $p \leqslant \psi(q)$. So consider $\phi(p) = \bigwedge\{q \in Q \mid p \leqslant \psi(q)\}$. Since ψ preserve meets, we have

$$p \leqslant \bigwedge\{\psi(q) \mid p \leqslant \psi(q)\} = \psi(\phi(p)) \text{ and } \phi(\psi(q)) = \bigwedge\{y \mid \psi(q) \leqslant \psi(y)\} \leqslant q$$

since $q \in \{y \mid \psi(q) \leqslant \psi(y)\}$. We can regard these inequalities as natural transformations $id_P \to \psi \centerdot \phi$ and $\phi \centerdot \psi \to id_Q$; so ϕ is left-adjoint of ψ. ■

1.5. **Adjointness of image and inverse image.** Let $\mathfrak{C}$ have $\mathcal{M}$-pullbacks and for every $f : X \to Y$ in $\mathfrak{C}$, let $f^{-1}(-) : \widehat{\mathcal{M}/Y} \to \widehat{\mathcal{M}/X}$ have a left adjoint

$$f(-) : \widehat{\mathcal{M}/X} \to \widehat{\mathcal{M}/Y}.$$

For $m : M \to X$ in $\widehat{\mathcal{M}/X}$, we call $f(m) : f(M) \to Y$ in $\widehat{\mathcal{M}/Y}$ the **image** of m under f; it is uniquely determined by the property

$$m \leqslant f^{-1}(n) \Leftrightarrow f(m) \leqslant n \tag{9}$$

for all $n \in \widehat{\mathcal{M}/Y}$. Furthermore, (2) yields to the following formulas

(1) $m \leqslant k \Rightarrow f(m) \leqslant (k)$;

(2) $m \leqslant f^{-1}(f(m))$ and $\left(f^{-1}(n)\right) \leqslant n$;

(3) $f\left(\bigvee_{i\in I} m_i\right) = \bigvee_{i\in I} f\left(m_i\right)$ and $f^{-1}\left(\bigwedge_{i\in I} n_i\right) = \bigwedge_{i\in I} f^{-1}\left(n_i\right)$.

Proposition 1.5. *When $\mathfrak{C}$ has $\mathcal{M}$-pullbacks and $(\mathcal{E}, \mathcal{M})$-factorization system for morphisms, we have*

(1) *If $f \in \mathcal{M}$, then $f^{-1}(o_Y) = o_X$ (provided the trivial subobject exists);*

(2) *$f \in \mathcal{E}$ if and only if $f(o_X) = o_Y$;*

(3) *If $f \in \mathcal{M}$, then $f^{-1}\left(f(m)\right) = m$ for all $m \in \widehat{\mathcal{M}/X}$;*

(4) *If $f \in \mathcal{E}$ and if $\mathcal{E}$ is stable under pullbacks, then $f\left(f^{-1}(n)\right) = n$ for all $n \in \widehat{\mathcal{M}/Y}$.*

Now,

Proposition 1.6. *Let $\mathfrak{C}$ be $\mathcal{M}$-complete and has $(\mathcal{E}, \mathcal{M})$-factorization system for morphisms, and assume the existence of an object P such that*

$$e \in \mathcal{E} \Leftrightarrow P \quad \textit{is projective with respect to} \quad e$$

holds for every morphism e in $\mathfrak{C}$. Then for a morphism $f : X \to Y$ and non-empty families $(m_i)_{i \,\in\, I}$ in $\widehat{\mathcal{M}/X}$ and $(n_i)_{i \,\in\, I}$ in $\widehat{\mathcal{M}/Y}$, we have:

(1) *If f is a monomorphism, then $f\left(\bigwedge_{i \in I} m_i\right) = \bigwedge_{i \in I} f(m_i)$;*

(2) *If the sink $(j_i : N_i \to N)_{i \in I}$ belonging to a union $n = \bigvee_{i \in I} n_i$ as in (6) has the property that for every $y : P \to N$ there is an $i \in I$ and a morphism $x : P \to N_i$ with $j_i \centerdot x = y$, then*

$$f^{-1}\left(\bigvee_{i \in I} n_i\right) = \bigvee_{i \in I} f^{-1}(n_i)$$

.

Proof. See [5], p. 23 ■

Observe that condition (2) of Proposition 1.6 and condition (4) of Proposition 1.4 imply that $f^{-1}(-) : \widehat{\mathcal{M}/Y} \to \widehat{\mathcal{M}/X}$ have a right adjoint

$$f_*(-) : \widehat{\mathcal{M}/X} \to \widehat{\mathcal{M}/Y}.$$

For $m : M \to X$ in $\widehat{\mathcal{M}/X}$; it is uniquely determined by the property

$$m \leqslant f^{-1}(n) \Leftrightarrow f_*(m) \leqslant n \tag{10}$$

for all $n \in \widehat{\mathcal{M}/Y}$. Furthermore, (1.2) implies that f_* is an order-preserving map.

2. Interior Operators

Throughout this section, we consider a category $\mathfrak{C}$ satisfying the conditions of Proposition 1.6.

Definition 2.1. *An interior operator I of the category $\mathfrak{C}$ with respect to the class $\mathcal{M}$ of subobjects is given by a family $I = (i_X)_{X \in \mathfrak{C}}$ of maps $i_X : \widehat{\mathcal{M}/X} \to \widehat{\mathcal{M}/X}$ such that*

- (I_1) *(Contraction) $i_X(m) \leqslant m$ for all $m \in \widehat{\mathcal{M}/X}$;*
- (I_2) *(Monotonicity) If $m \leqslant k$ in $\widehat{\mathcal{M}/X}$, then $i_X(m) \leqslant i_X(k)$*
- (I_3) *(Upper bound) $i_X(1_X) = 1_X$.*

Definition 2.2. *An I-space is a pair (X, i_X) where X is an object of $\mathfrak{C}$ and i_X is an interior operator on X.*

Definition 2.3. *A morphism $f : X \to Y$ of $\mathfrak{C}$ is said to be I-continuous if*

$$f^{-1}\left(i_Y(m)\right) \leqslant i_X\left(f^{-1}(m)\right) \tag{11}$$

for all $m \in \widehat{\mathcal{M}/Y}$.

Proposition 2.4. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of $\mathfrak{C}$ I-continuous then $g \centerdot f$ is a morphism of $\mathfrak{C}$ which is I-continuous.*

Proof. Since $g : Y \to Z$ is I-continuous, we have $g^{-1}\big(i_Z(m)\big) \leqslant i_Y\big(g^{-1}(m)\big)$ for all $m \in \widehat{\mathcal{M}/Z}$, it fallows that

$$f^{-1}\Big(g^{-1}\big(i_Z(m)\big)\Big) \leqslant f^{-1}\Big(i_Y\big(g^{-1}(m)\big)\Big);$$

now, by the I-continuity of f,

$$f^{-1}\Big(i_Y\big(g^{-1}(m)\big)\Big) \leqslant i_X\Big(f^{-1}\big(g^{-1}(m)\big)\Big),$$

threfore

$$f^{-1}\Big(g^{-1}\big(i_Z(m)\big)\Big) \leqslant i_X\Big(f^{-1}\big(g^{-1}(m)\big)\Big),$$

that is to say

$$(g \centerdot f)^{-1}\big(i_Z(m))\big) \leqslant i_X\Big((g \centerdot f)^{-1}(m)\Big)$$

for all $m \in \widehat{\mathcal{M}/Z}$. This complete the proof. ■

As a consequence we obtain

Definition 2.5. *The category $\mathfrak{C}_I$ of I-spaces comprises the following data:*

(1) **Objects**: *Pairs (X, i_X) where X is an object of $\mathfrak{C}$ and i_X is an interior operator on X.*

(2) **Morphisms**: *Morphisms of $\mathfrak{C}$ which are I-continuous.*

2.1. **The lattice structure of all interior operators.** For a category $\mathfrak{C}$ satisfying the conditions of Proposition 1.6 we consider the conglomerate

$$Int(\mathfrak{C}, \mathcal{M})$$

of all interior operators on $\mathfrak{C}$ with respect to $\mathcal{M}$. It is ordered by

$$I \leqslant J \Leftrightarrow i_X(n) \leqslant j_X(n), \quad \text{for all } n \in \widehat{\mathcal{M}/X} \text{ and all } X \text{ object of } \mathfrak{C}.$$

This way $Int(\mathfrak{C}, \mathcal{M})$ inherits a lattice structure from $\mathcal{M}$:

Proposition 2.6. *For $\mathfrak{C}$ $\mathcal{M}$-complete, every family $(I_\lambda)_{\lambda \in \Lambda}$ in $Int(\mathfrak{C}, \mathcal{M})$ has a join $\bigvee\limits_{\lambda \in \Lambda} I_\lambda$ and a meet $\bigwedge\limits_{\lambda \in \Lambda} I_\lambda$ in $Int(\mathfrak{C}, \mathcal{M})$. The discrete interior operator*

$$I_D = (i_{D\,X})_{X \in \mathfrak{C}} \;\; \textit{with} \;\; i_{D\,X}(m) = m \;\; \textit{for all} \;\; m \in \widehat{\mathcal{M}/X}$$

is the largest element in $Int(\mathfrak{C}, \mathcal{M})$, and the trivial interior operator

$$I_T = (i_{T\,X})_{X \in \mathfrak{C}} \;\; \textit{with} \;\; i_{T\,X}(m) = \begin{cases} o_X & \textit{for all } m \in \widehat{\mathcal{M}/X}, \;\; m \neq 1_X \\ 1_X & \textit{if } m = 1_X \end{cases}$$

is the least one.

Proof. For $\Lambda \neq \emptyset$, let $\widetilde{I} = \bigvee\limits_{\lambda \in \Lambda} I_\lambda$, then

$$\widetilde{i_X} = \bigvee_{\lambda \in \Lambda} i_{\lambda X},$$

for all X object of $\mathfrak{C}$, satisfies

- $\widetilde{i_X}(m) \leqslant m$, because $i_{\lambda X}(m) \leqslant m$ for all $m \in \widehat{\mathcal{M}/X}$ and for all $\lambda \in \Lambda$.
- If $m \leqslant k$ in $\widehat{\mathcal{M}/X}$ then $i_{\lambda X}(m) \leqslant i_{\lambda X}(k)$ for all $m \in \widehat{\mathcal{M}/X}$ and for all $\lambda \in \Lambda$, threfore $\widetilde{i_X}(m) \leqslant \widetilde{i_X}(k)$.
- Since $i_{\lambda X}(1_X) = 1_X$ for all $m \in \widehat{\mathcal{M}/X}$ and for all $\lambda \in \Lambda$, we have that $\widetilde{i_X}(1_X) = 1_X$.

Similarly $\bigwedge\limits_{\lambda \in \Lambda} I_\lambda$, I_D and I_T are interior operators. ■

Corollary 2.7. *For $\mathfrak{C}$ $\mathcal{M}$-complete and for every object X of $\mathfrak{C}$*

$$Int(X) = \{i_X \mid i_X \quad \textit{is an interior operator on } X\}$$

is a complete lattice.

2.2. **Initial interior operators.** Let $\mathfrak{C}$ be a category satisfying the conditions of Proposition 1.6, let (Y, i_Y) be an object of $\mathfrak{C}_I$ and let X be an object of $\mathfrak{C}$. For each morphism $f : X \to Y$ in $\mathfrak{C}$ we define on X the operotor

$$i_{X_f} := f^{-1} \centerdot i_Y \centerdot f_*. \tag{12}$$

Proposition 2.8. *The operator (12) ia an interior operator on X for which the morphism f is I-continuous.*

Proof.

(I_1) (Contraction) $i_{X_f}(m) = f^{-1} \centerdot i_Y \centerdot f_*(m) \leqslant f^{-1} \centerdot f_*(m) \leqslant m$ for all $m \in \widehat{\mathcal{M}/X}$;

(I_2) (Monotonicity) $m \leqslant k$ in $\widehat{\mathcal{M}/X}$, implies $f_*(m) \leqslant f_*(k)$, then $i_Y \centerdot f_*(m) \leqslant i_Y \centerdot f_*(k)$, consequently $f^{-1} \centerdot i_Y \centerdot f_*(m) \leqslant f^{-1} \centerdot i_Y \centerdot f_*(k)$;

(I_3) (Upper bound) $i_{X_f}(1_X) = f^{-1} \centerdot i_Y \centerdot f_*(1_X) = 1_X$.

Finally,

$$\begin{aligned} f^{-1}\big(i_Y(n)\big) &\leqslant f^{-1}\big(i_Y \centerdot f_* \centerdot f^{-1}(n)\big) = (f^{-1} \centerdot i_Y \centerdot f_*)\big(f^{-1}(n)\big) \\ &= i_{X_f}\big(f^{-1}(n)\big), \end{aligned}$$

for all $n \in \widehat{\mathcal{M}/Y}$. ∎

It is clear that i_{X_f} is the coarsest interior operator on X for which the morphism f is I-continuous; more precisaly

Proposition 2.9. *Let (Z, i_Z) and (Y, i_Y) be objects of $\mathfrak{C}_I$, and let X be an object of $\mathfrak{C}$. For each morphism $g : Z \to X$ in $\mathfrak{C}$ and for $f : (X, i_{X_f}) \to (Y, i_Y)$ an I-continuous morphism, g is I-continuous if and only if $g \centerdot f$ is I-continuous.*

Proof. Suppose that $g \centerdot f$ is I-continuous, i. e.

$$(f \centerdot g)^{-1}\big(i_Y(n)\big) \leqslant i_Z\big((f \centerdot g)^{-1}(n)\big)$$

for all $n \in \widehat{\mathcal{M}/Y}$. Then, for all $m \in \widehat{\mathcal{M}/X}$, we have

$$\begin{aligned} g^{-1}\big(i_{X_f}(m)\big) &= g^{-1}\big(f^{-1} \centerdot i_Y \centerdot f_*(m)\big) = (f \centerdot g)^{-1}\big(i_Y(f_*(m))\big) \\ &\leqslant i_Z\big((f \centerdot g)^{-1}(f_*(m))\big) = i_Z\big(g^{-1} \centerdot f^{-1} \centerdot f_*(m)\big) \\ &\leqslant i_Z\big(g^{-1}(m)\big). \end{aligned}$$

∎

As a consequence of corollary(2.7), proposition(2.8) and proposition (2.9) (cf. [1] or [11]), we obtain

Theorem 2.10. *Let $\mathfrak{C}$ be an $\mathcal{M}$-complete category then the concrete category $(\mathfrak{C}_I, U)$ over $\mathfrak{C}$ is a topological category.*

2.3. **Open subobjects.**

Definition 2.11. *An $\mathcal{M}$-subobject $m : M \to X$ is called I-open (in X) if it is isomorphic to its I-interior, that is: if $j_m : i_X(M) \to M$ is an isomorphism.*

The I-continuity condition (11) implies that I-openness is preserve by inverse images:

Proposition 2.12. *Let $f : X \to Y$ be a morphism in $\mathfrak{C}$. If n is I-open in Y, then $f^{-1}(n)$ is I-open in X.*

Proof. If $n \cong i_Y(n)$ then $f^{-1}(n) = f^{-1}\big(i_Y(n)\big) \leqslant i_X\big(f^{-1}(n)\big)$, so $i_X\big(f^{-1}(n)\big) \cong f^{-1}(n)$. ∎

Let $\mathcal{M}^I$ be the class of I-open $\mathcal{M}$-subobjects. The last proposition asserts that $\mathcal{M}^I$ is stable under pullback, therefore

Corollary 2.13. *If, for monomorphisms m and n, $n \cdot m$ is an I-open $\mathcal{M}$-subobject, then m is an I-open $\mathcal{M}$-subobject.*

3. Examples of Interior Operators

3.1. **Kuratowski interior operator.** The Kuratowski interior operator $I = (i_X)_{X \in Sets}$ is described as follows (cf. [6]):

Definition 3.1. *Let X be a set and $i_X : \wp(X) \to \wp(X)$ a map such that:*

(1) $i_X(X) = X$.
(2) $i_X(A) \subseteq A$ *for all* $A \in \wp(X)$.
(3) $i_X \cdot i_X(A) = i_X(A)$ *for all* $A \in \wp(X)$.
(4) $i_X(A \cap B) = i_X(A) \cap i_X(B)$ *for all* $A, B \in \wp(X)$.

Then $\tau = \{A \in \wp(X) \mid i_X(A) = A\}$ is a topology on X.

3.2. **Grothendieck interior operator.** Let $\mathfrak{C}$ be a small category, and let $\mathbf{Sets}^{\mathfrak{C}^{\mathbf{op}}}$ be the corresponding functor category (cf. [10]). As usual, we write

$$y : \mathfrak{C} \to \mathbf{Sets}^{\mathfrak{C}^{\mathbf{op}}}$$

for the Yoneda embedding: $y(C) = Hom_{\mathfrak{C}}(-, C)$. Recall that

(1) A sieve S on C is a subobject $S \subseteq y(C)$ in $\mathbf{Sets}^{\mathfrak{C}^{\mathbf{op}}}$. We write $Sub\ y(C)$ for the class of subobjects of $y(C)$.

(2) A sieve S on C is a right ideal of morphisms in $\mathfrak{C}$, all with codomain C.

(3) If S is a sieve on C and $h : D \to C$ is any arrow to C, then

$$h^*(S) = \{g \mid cod(g) = D \;\; h \centerdot g \in S\}$$

is a sieve on D.

(4) $t_C = \{f \mid cod(f) = C\}$ is the maximal sieve on C

Definition 3.2. *An interior operator I of the category $\mathfrak{C}$ is given by a family $I = (i_{y(C)})_{C \in \mathfrak{C}}$ of maps $i_{y(C)} : Sub\ y(C) \to Sub\ y(C)$ such that*

(I_1) *(Contraction) $i_{y(C)}(S) \leqslant S$ for all $S \in Sub\ y(C)$;*

(I_2) *(Monotonicity) If $S_1 \leqslant S_2$ in $Sub\ y(C)$, then $i_{y(C)}(S_1) \leqslant i_{y(C)}(S_2)$*

(I_3) *(Upper bound) $i_{y(C)}(t_C) = t_C$.*

Proposition 3.3. *Suppose that $\mathfrak{C}$ have $(\mathcal{E}, \mathcal{M})$-factorization with $\mathcal{M}$-pullbacks, and $\mathcal{E}$ is stable under pullbacks. Then the function J which assigns to each object C of $\mathfrak{C}$ the collection $J(C) = \{S \mid S \text{ is } I\text{-open}\}$ is a Grothendieck topology on $\mathfrak{C}$, whenever there exists an $\mathcal{E}$-morphisms in each sieve S.*

Proof.

(1) Clearly, $t_C \in J(C)$.

(2) Suppose that $S \in J(C)$ and $h : D \to C$ is any arrow to C. Then for

$$i_{y(D)_h} = h^* \centerdot i_{y(C)} \centerdot h_*,$$

we have

$$i_{y(D)_h}\big(h^*(S)\big) = h^* \centerdot i_{y(C)} \centerdot h_*\big(h^*(S)\big) \geqslant h^* \centerdot i_{y(C)} \centerdot (S) = h^*(S),$$

consequently, $h^*(S) \in J(D)$.

(3) Let S be in $J(C)$, and let R be any sieve on C such that $h^*(R) \in J(D)$ for all $h : D \to C$ in S. Since there exists an $\mathcal{E}$-morphisms g in S, and since $g_* \centerdot g^*(R) \cong R$, it follows that

$$R \cong g_*\left(g^*(R)\right) \cong g_*\left(i_{y(D)}\left(g^*(R)\right)\right) = g_*\left(g^* \centerdot i_{y(C)} \centerdot g_*\left(g^*(R)\right)\right) \cong i_{y(C)}(R).$$

■

3.3. **Interior operators on Grothendieck topos.** Recall that a Grothendieck topos is a category which is equivalent to the category $Sh(\mathfrak{C}, J)$ of sheaves on some site $(\mathfrak{C}, J)$ (cf. [10]). Furthermore, for any sheaf E on a site $(\mathfrak{C}, J)$, the lattice $Sub\ (E)$ of all subsheaves of E is a complete Heyting algebra. It is also true that any morphism $\phi : E \to F$ of sheaves on a site induces a functor on the corresponding partially ordered sets of subsheaves,

$$\phi^{-1} : Sub(F) \to Sub(E) \tag{13}$$

by pullback. Moreover, this functor has both a left and a right adjoint:

$$\exists_\phi \dashv \phi^{-1} \dashv \forall_\phi. \tag{14}$$

Definition 3.4. *An interior operator I of the category $Sh(\mathfrak{C}, J)$ of sheaves on some site $(\mathfrak{C}, J)$ is given by a family $I = (i_E)_{E \in Sh(\mathfrak{C}, J)}$ of maps $i_E : Sub\ E \to Sub\ E$ such that*

(I_1) *(Contraction) $i_E(A) \leqslant A$ for all $A \in Sub\ E$;*

(I_2) *(Monotonicity) If $A \leqslant B$ in $Sub\ E$, then $i_E(A) \leqslant i_E(B)$*

(I_3) (*Upper bound*) $i_E(E) = E$.

As a consequence we have a category $Sh(\mathfrak{C}, J)I$ whose objects are pairs (E, i_E) where E is a sheave on the site $(\mathfrak{C}, J)$, and whose morphisms are morphisms of $Sh(\mathfrak{C}, J)$ which are I-continuous; i. e., morphisms $\phi : E \to F$ such that

$$\phi^{-1}\left(i_F(B)\right) \leqslant i_E\left(\phi^{-1}(B)\right)$$

for all $B \in Sub\ E$.

Given an object (F, i_F) of $Sh(\mathfrak{C}, J)I$ and a morphism $\phi : E \to F$ of $Sh(\mathfrak{C}, J)I$, defining

$$i_{E_\phi} := \phi^{-1} \centerdot i_F \centerdot \forall_\phi,$$

it is clear that

Proposition 3.5. *The concrete category $(Sh(\mathfrak{C}, J)_I, U)$ over $Sh(\mathfrak{C}, J)$ is a topological category.*

3.4. **Interior operators on the category LF-Top of fuzzy topological spaces.** Given a GL-monoid $(L, \leqslant, \otimes)$ (for example, a complete Heyting algebra or a complete MV- algebra), for any set X (cf:[7]),

Definition 3.6. *A mapping $\mathcal{I} : L^X \times L \to L^X$ is called an L-fuzzy interior operator on X if and only if $\mathcal{I}$ satisfies the following conditions:*

(I_1) $\mathcal{I}(1_X, \alpha) = 1_X$, *for all* $\alpha \in L$.

(I_2) $\mathcal{I}(g, \beta) \leqslant \mathcal{I}(f, \alpha)$ *whenever* $g \leqslant f$ *and* $\alpha \leqslant \beta$.

(I_3) $\mathcal{I}(f, \alpha) \otimes \mathcal{I}(g, \beta) \leqslant \mathcal{I}(f \otimes g, \alpha \otimes \beta)$.

(I_4) $\mathcal{I}(f, \alpha) \leqslant f$.

(I_5) $\mathcal{I}(f, \alpha) \leqslant \mathcal{I}\big(\mathcal{I}(f, \alpha)\big)$.

(I_6) $\mathcal{I}(f, \perp) = f$.

(I_7) *If* $\emptyset \neq K \subseteq L$ *and* $\mathcal{I}(f, \alpha) = f^0$, *then* $\mathcal{I}(f, \bigvee K) = f^0$.

Given an L-fuzzy interior operator $\mathcal{I} : L^X \times L \to L^X$, the formula

$$\mathcal{T}_{\mathcal{I}}(f) = \bigvee\{\alpha \in L \mid f \leqslant \mathcal{I}(f, \alpha)\}, \quad f \in L^X,$$

defines an L-fuzzy topology $\mathcal{T}_{\mathcal{I}} : L^X \to L$ on X.

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